

# Finiteness of the basic intersection cohomology of a Killing foliation

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**Abstract** We prove that the basic intersection cohomology  $\mathbb{H}_p^*(M/\mathcal{F})$ , where  $\mathcal{F}$  is the singular foliation determined by an isometric action of a Lie group  $G$  on the compact manifold  $M$ , is finite dimensional.

**Keywords** Basic intersection cohomology · Lie group actions

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## 1 Introduction

This paper deals with an action  $\Phi: G \times M \rightarrow M$  of a Lie group on a compact manifold preserving a riemannian metric on it. The orbits of this action define a singular foliation  $\mathcal{F}$  on  $M$ . Putting together the orbits of the same dimension we get a stratification of  $M$ . This structure is still very regular. The foliation  $\mathcal{F}$  is in fact a conical foliation and we can define the basic intersection cohomology  $\mathbb{H}_p^*(M/\mathcal{F})$  (cf. [10]). This invariant becomes the basic cohomology  $H^*(M/\mathcal{F})$  when the action  $\Phi$  is almost free, and the intersection cohomology  $\mathbb{H}_p^*(M/G)$  when the Lie group  $G$  is compact.

The aim of this work is to prove that this cohomology  $\mathbb{H}_p^*(M/\mathcal{F})$  is finite dimensional. This result generalizes [3] (almost free case), [11] (abelian case) and [10] (compact case).

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The paper is organized as follows. In Sect. 2 we present the foliation  $\mathcal{F}$ . The basic intersection cohomology  $\mathbb{H}_p^*(M/\mathcal{F})$  associated to this foliation is studied in Sect. 3. Twisted products are studied in Sect. 4. The finiteness of  $\mathbb{H}_p^*(M/\mathcal{F})$  is proved in Sect. 5.

In the sequel  $M$  is a connected, second countable, Hausdorff, without boundary and smooth (of class  $C^\infty$ ) manifold of dimension  $m$ . All the maps are considered smooth unless something else is indicated.

## 2 Killing foliations determined by isometric actions

We study in this work the foliations induced by isometric actions: the *Killing foliations*. These foliations are examples of the conical foliations for which the basic intersection cohomology has been defined (see [10, 11]). We present this geometrical framework in this section.

### 2.1 Killing foliations

A smooth action  $\Phi: G \times M \rightarrow M$  of a Lie group  $G$  on a manifold  $M$  is a *isometric action* when there exists a riemannian metric  $\mu$  on  $M$  preserved by  $G$ .

The connected components of the orbits of the action  $\Phi$  determine a partition  $\mathcal{F}$  on  $M$ . In fact, this partition is a singular riemannian foliation that we shall call *Killing foliation* (cf. [7]). Notice that  $\mathcal{F}$  is also a conical foliation in the sense of [10, 11]. Classifying the points of  $M$  following the dimension of the leaves of  $\mathcal{F}$  one gets the *stratification*  $\mathbf{S}_{\mathcal{F}}$  of  $\mathcal{F}$ . It is determined by the equivalence relation  $x \sim y \Leftrightarrow \dim G_x = \dim G_y$ . The elements of  $\mathbf{S}_{\mathcal{F}}$  are called *strata*.

In the particular case where the closure of  $G$  in the isometry group of  $(M, \mu)$  is a compact Lie group<sup>1</sup> we shall say that the action  $\Phi$  is a *tame action*. In fact, a smooth action  $\Phi: G \times M \rightarrow M$  is tame if and only if it extends to a smooth action  $\Phi: K \times M \rightarrow M$  where  $K$  is a compact Lie group containing  $G$  (cf. [6]). The group  $K$  is not unique, but we always can choose  $K$  in such a way that  $G$  is dense in  $K$ . We shall say that  $K$  is a *tamer group*. Here the strata of  $\mathbf{S}_{\mathcal{F}}$  are  $K$ -invariant closed submanifolds of  $M$ .

Since the aim of this work is the study of  $\mathcal{F}$  and not the action  $\Phi$  itself, we can consider that the Lie group  $G$  is connected. Let us see that.

**Proposition 1** *Let  $\Phi: G \times M \rightarrow M$  is a tame action. Let  $G_0$  be the connected component of  $G$  containing the unity element. The Killing foliation defined by the restriction  $\Phi: G_0 \times M \rightarrow M$  is also  $\mathcal{F}$ .*

*Proof* The partition  $\mathcal{F}$  is defined by this equivalence relation:

$$x \sim y \iff \exists \text{ continuous path } \alpha: [0, 1] \rightarrow G(x) \text{ such that } \alpha(0) = x \text{ and } \alpha(1) = y.$$

Since the map  $\Delta: G \rightarrow G(x)$ , defined by  $\Delta(g) = \Phi(g, x) = g \cdot x$ , is a submersion (see for example [2]) then

$$x \sim y \iff \exists \text{ continuous path } \beta: [0, 1] \rightarrow G \text{ such that } \beta(0) = e \text{ and } \beta(1) \cdot x = y,$$

and by definition of  $G_0$

$$x \sim y \iff \exists \text{ continuous path } \beta: [0, 1] \rightarrow G_0 \text{ such that } \beta(0) = e \text{ and } \beta(1) \cdot x = y.$$

This gives the result.  $\square$

<sup>1</sup> This is always the case when the manifold  $M$  is a compact.

When  $G$  is connected, the tamer group  $K$  has richer properties.

**Proposition 2** *Let  $G$  be a connected Lie subgroup of a compact Lie group  $K$ . If  $G$  is dense in  $K$  then  $G \triangleleft K$  and the quotient group  $K/G$  is commutative.*

*Proof* The Lie algebra  $\mathfrak{g}$  is  $\text{Ad}_G$ -invariant and hence, by density,  $\text{Ad}_K$ -invariant. Then  $\mathfrak{g}$  is an ideal of  $\mathfrak{k}$ . The connectedness of  $G$  gives that  $G$  is a normal subgroup of  $K$ . Since  $\text{Ad}_G$  acts trivially on  $\mathfrak{k}/\mathfrak{g}$ ,  $\text{Ad}_K$  acts trivially, too. Therefore,  $\mathfrak{k}/\mathfrak{g}$  is abelian (see for example [8, p. 628]).  $\square$

## 2.2 Particular tame actions

A *trio* is a triple  $(K, G, H)$ , with  $K$  a compact Lie group,  $G$  a normal subgroup of  $K$  and  $H$  a closed subgroup of  $K$ . We present now some tame actions associated to a trio  $(K, G, H)$ . They are going to be intensively used in this work. First of all we need some definitions.

- The action  $\Phi_l: K \times K \rightarrow K$  is defined by  $\Phi_l(g, k) = g \cdot k$ . For each element  $u$  of the Lie algebra  $\mathfrak{k}$  of  $K$ , we shall write  $X^u$  the associated (right invariant) vector field. It is defined by  $X^u(k) = T_e R_k(u)$  where  $R_k: K \rightarrow K$  is given by  $R_k(\ell) = \ell \cdot k$ .
- The action  $\Phi_r: K \times K \rightarrow K$  is defined by  $\Phi_r(g, k) = k \cdot g^{-1}$ . For each element  $u \in \mathfrak{k}$  of  $K$ , we shall write  $X_u$  the associated (left invariant) vector field. It is defined by  $X_u(k) = -T_e L_k(u)$  where  $L_k: K \rightarrow K$  is given by  $L_k(\ell) = k \cdot \ell$ .
- The action  $\Psi: K \times K/H \rightarrow K/H$  is defined by  $\Psi(g, kH) = (g \cdot k)H$ . For each element  $u \in \mathfrak{k}$ , we shall write  $Y_u$  the associated vector field. Since the canonical projection  $\pi: K \rightarrow K/H$  is a  $K$ -equivariant map, then we have  $\pi_* X^u = Y_u$  for each  $u \in \mathfrak{k}$ .
- The action  $\Gamma: H \times H \rightarrow H$  is defined by  $\Gamma(g, h) = g \cdot h$ . For each element  $u$  of the Lie algebra  $\mathfrak{h}$  of  $H$  we write  $Z^u$  the associated (right invariant) vector field.

The associated actions we are going to use are the following.

(a) *The restriction  $\Phi_l: G \times K \rightarrow K$ , which induces the regular Killing foliation  $\mathcal{K}$ .*

(b) *The restriction  $\Phi_r: G \times K \rightarrow K$ , which induces the regular Killing foliation  $\mathcal{K}$ .*

Since  $G \triangleleft K$ , the foliation  $\mathcal{K}$  is determined by the family of vector fields  $\{X^u / u \in \mathfrak{g}\}$ , where  $\mathfrak{g}$  is the Lie algebra of  $G$ , and also by the family  $\{X_u / u \in \mathfrak{g}\}$ . The orbits  $G(k) = Gk = kG$  have the same dimension  $\dim G$ .

(c) *The restriction  $\Psi: G \times K/H \rightarrow K/H$ , which induces the regular Killing foliation  $\mathcal{D}$ .*

The foliation  $\mathcal{D}$  is determined by the family of vector fields  $\{Y_u / u \in \mathfrak{g}\}$ . The orbits  $G(kH)$  have the same dimension  $\dim G - \dim(G \cap H)$ .

(d) *The restriction  $\Gamma: (G \cap H) \times H \rightarrow H$ , which induces the regular Killing foliation  $\mathcal{C}$ .*

The foliation  $\mathcal{C}$  is determined by the family of vector fields  $\{Z^u / u \in \mathfrak{g} \cap \mathfrak{h}\}$ . The orbits  $(G \cap H)(k)$  have the same dimension  $\dim(G \cap H)$ .

(e) *The restriction  $\Phi_r: GH \times K \rightarrow K$ , which induces the regular Killing foliation  $\mathcal{E}$ .*

Notice that  $GH$  is a Lie group since  $G$  is normal in  $K$ . The foliation  $\mathcal{E}$  is, in fact, determined by the vector fields  $\{X_u / u \in \mathfrak{g} + \mathfrak{h}\}$ . The orbits  $(GH)(k)$  have the same dimension  $\dim G + \dim H - \dim(G \cap H)$ .

### 2.3 Twisted product

In order to prove the finiteness of the basic intersection cohomology we decompose the manifold in a finite number of simpler pieces. These are the twisted products we introduce now.

We fix a trio  $(K, G, H)$  and a smooth action  $\Theta: H \times N \rightarrow N$  of  $H$  on the manifold  $N$ . The *twisted product* is the quotient  $K \times_H N$  of  $K \times N$  by the equivalence relation  $(k, z) \sim (k \cdot h^{-1}, \Theta(h, z) = h \cdot z)$ . The element of  $K \times_H N$  corresponding to  $(k, z) \in K \times N$  is denoted by  $\langle k, z \rangle$ . This manifold is endowed with the tame action

$$\Phi: G \times (K \times_H N) \longrightarrow (K \times_H N),$$

defined by  $\Phi(g, \langle k, z \rangle) = \langle g \cdot k, z \rangle$ . We denote by  $\mathcal{W}$  the induced Killing foliation.

We also use the following tame action, namely, the restriction

$$\Theta: (G \cap H) \times N \rightarrow N$$

whose induced Killing foliation is denoted by  $\mathcal{N}$ .

The canonical projection  $\Pi: K \times N \rightarrow K \times_H N$  relates the involved foliations as follows:

(a)  $\Pi_*(\mathcal{K} \times \mathcal{I}) = \mathcal{W}$ , where  $\mathcal{I}$  is the pointwise foliation (since the map  $\Pi$  is  $G$ -equivariant).

(b)  $\mathcal{S}_{\mathcal{W}} = \{\Pi(K \times S) / S \in \mathcal{S}_{\mathcal{N}}\} = \Pi(\{K\} \times \mathcal{S}_{\mathcal{N}})$  (since  $G_{\langle k, z \rangle} = k(G \cap H)_z k^{-1}$ ).

### 3 Basic intersection cohomology

In this section we recall the definition of the basic intersection<sup>2</sup> cohomology and we present the main properties we are going to use in this work. For the rest of this section, we fix a conical foliation  $\mathcal{F}$  defined on a manifold  $M$ . The associated stratification is  $\mathcal{S}_{\mathcal{F}}$ . The regular stratum of  $\mathcal{F}$  is denoted by  $R_{\mathcal{F}}$ . We shall write  $m = \dim M$ ,  $r = \dim \mathcal{F}$  and  $s = m - r = \text{codim}_M \mathcal{F}$ .

We are going to deal with differential forms on a product (manifold)  $\times [0, 1]^p$ , they are restrictions of differential forms defined on (manifold)  $\times ]-1, 1[^p$ .

#### 3.1 Perverse forms

Recall that a *conical chart* is a foliated diffeomorphism  $\varphi: (\mathbb{R}^{m-n-1} \times c\mathbb{S}^n, \mathcal{H} \times c\mathcal{G}) \rightarrow (U, \mathcal{F}_U)$  where  $(\mathbb{R}^{m-n-1}, \mathcal{H})$  is a simple foliation and  $(\mathbb{S}^n, \mathcal{G})$  is a conical foliation without 0-dimensional leaves. We also shall denote this chart by  $(U, \varphi, S)$  where  $S$  is the stratum of  $\mathcal{S}_{\mathcal{F}}$  verifying  $\varphi(\mathbb{R}^{m-n-1} \times \{\vartheta\}) = U \cap S$ .

The differential complex  $\Pi_{\mathcal{F}}^*(M \times [0, 1]^p)$  of *perverse forms* of  $M \times [0, 1]^p$  is introduced by induction on depth  $\mathcal{S}_{\mathcal{F}}$ . When this depth is 0 then

$$\Pi_{\mathcal{F}}^*(M \times [0, 1]^p) = \Omega^*(M \times [0, 1]^p).$$

Consider now the generic case. A perverse form of  $M \times [0, 1]^p$  is first of all a differential form  $\omega \in \Omega^*(R_{\mathcal{F}} \times [0, 1]^p)$  such that,

$$\begin{cases} \text{the pull-back } (\varphi \times \mathbb{I}_{[0, 1]^p})^* \omega \in \Omega^*(\mathbb{R}^{m-n-1} \times R_{\mathcal{G}} \times ]0, 1[ \times [0, 1]^p) \\ \text{extends to } \omega_{\varphi} \in \Pi_{\mathcal{H} \times c\mathcal{G}}^*(\mathbb{R}^{m-n-1} \times \mathbb{S}^n \times [0, 1]^{p+1}) \end{cases}$$

<sup>2</sup> We refer the reader to [10, 11] for details.

for any conical chart  $(U, \varphi)$ , where  $\mathbb{I}_\bullet$  stands for the identity map. Notice that  $\Omega^*(M)$  is included on  $\Pi_{\mathcal{F}}^*(M)$ <sup>3</sup>.

### 3.2 Perverse degree

The amount of transversality of a perverse form  $\omega \in \Pi_{\mathcal{F}}^*(M)$  with respect to a singular stratum  $S \in \mathcal{S}_{\mathcal{F}}$  is measured by the perverse degree  $\|\omega\|_S$ . We recall here the definition of local perverse degree  $\|\omega\|_U \in \{-\infty\} \cup \mathbb{N}$  of  $\omega$  relatively to a conical chart  $(U, \varphi, S)$ :

1.  $\|\omega\|_U = -\infty$  when  $\omega_\varphi \equiv 0$  on  $\mathbb{R}^{m-n-1} \times R_G \times \{0\}$ ,
2.  $\|\omega\|_U \leq p$ , with  $p \in \mathbb{N}$ , when  $\omega_\varphi(v_0, \dots, v_p, -) \equiv 0$  where the vectors  $\{v_0, \dots, v_p\}$  are tangent to the fibers of  $P_\varphi : \mathbb{R}^{m-n-1} \times R_G \times \{0\} \longrightarrow U \cap S$ <sup>4</sup>.

This number does not depend on the choice of the conical chart (cf. [11, Proposition 1.3.1]). Finally, we define the *perverse degree*  $\|\omega\|_S$  by

$$\|\omega\|_S = \sup \{ \|\omega\|_U \mid (U, \varphi, S) \text{ conical chart} \}.$$

The perverse degree of  $\omega \in \Omega^*(M)$  verifies  $\|\omega\|_S \leq 0$  for any singular stratum  $S \in \mathcal{S}_{\mathcal{F}}$  (cf. 3.1).

### 3.3 Basic cohomology

The basic cohomology of the foliation  $\mathcal{F}$  is an important tool to study its transversal structure and plays the role of the cohomology of the orbit space  $M/\mathcal{F}$ , which can be a wild topological space. A differential form  $\omega \in \Omega^*(M)$  is *basic* if  $i_X \omega = i_X d\omega = 0$ , for each vector field  $X$  on  $M$  tangent to the foliation  $\mathcal{F}$ . For example, a function  $f$  is basic iff  $f$  is constant on the leaves of  $\mathcal{F}$ . We shall write  $\Omega^*(M/\mathcal{F})$  for the complex of basic forms. Its cohomology  $H^*(M/\mathcal{F})$  is the *basic cohomology* of  $(M, \mathcal{F})$ . We also use the *relative basic cohomology*  $H^*((M, F)/\mathcal{F})$ , that is, the cohomology computed from the complex of basic forms vanishing on the saturated set  $F \subset M$ . The basic cohomology does not use the stratification  $\mathcal{S}_{\mathcal{F}}$ .

### 3.4 Basic intersection cohomology

A *perversity* is a map  $\bar{p} : \mathcal{S}_{\mathcal{F}}^\sigma \rightarrow \mathbb{Z} \cup \{-\infty, \infty\}$ , where  $\mathcal{S}_{\mathcal{F}}^\sigma$  is the family of singular strata. The *constant perversity*  $\bar{\iota}$  is defined by  $\bar{\iota}(S) = \iota$ , where  $\iota \in \mathbb{Z} \cup \{-\infty, \infty\}$ .

The basic intersection cohomology appears when one considers basic perverse forms whose perverse degree is controlled by a perversity. We shall put

$$\Omega_{\bar{p}}^*(M/\mathcal{F}) = \left\{ \omega \in \Pi_{\mathcal{F}}^*(M) \mid \omega \text{ is basic and } \max(\|\omega\|_S, \|d\omega\|_S) \leq \bar{p}(S) \quad \forall S \in \mathcal{S}_{\mathcal{F}}^\sigma \right\}$$

the complex of basic perverse forms whose perverse degree (and that of their derivative) is bounded by the perversity  $\bar{p}$ . The cohomology  $\mathbb{H}_{\bar{p}}^*(M/\mathcal{F})$  of this complex is the *basic intersection cohomology*<sup>5</sup> of  $(M, \mathcal{F})$  relatively to the perversity  $\bar{p}$ .

<sup>3</sup> Through the restriction  $\omega \mapsto \omega_{R_{\mathcal{F}}}$ .

<sup>4</sup> The map  $P_\varphi : \mathbb{R}^{m-n-1} \times \mathbb{S}^n \times [0, 1] \longrightarrow U$  is defined by  $P_\varphi(x, y, t) = \varphi(x, [y, t])$ .

<sup>5</sup> BIC for short.

Consider a twisted product  $K \times_H N$ . Perversities on  $K \times_H N$  and  $K \times N$  are determinate by perversities on  $N$  by the formula (cf. 2.3 (b)):

$$\overline{p}(K \times S) = \overline{p}(\Pi(K \times S)) = \overline{p}(S). \quad (1)$$

### 3.5 Mayer-Vietoris

This is the technique we use in order to decompose the manifold in nicer pieces. An open covering  $\{U, V\}$  of  $M$  by saturated open subsets is a *basic covering*. It possesses a subordinated partition of the unity made up of basic functions defined on  $M$  (see [9]). For a such covering we have the Mayer-Vietoris short sequence

$$0 \rightarrow \Omega_{\overline{p}}^*(M/\mathcal{F}) \rightarrow \Omega_{\overline{p}}^*(U/\mathcal{F}) \oplus \Omega_{\overline{p}}^*(V/\mathcal{F}) \rightarrow \Omega_{\overline{p}}^*((U \cap V)/\mathcal{F}) \rightarrow 0,$$

where the map are defined by  $\omega \mapsto (\omega, \omega)$  and  $(\alpha, \beta) \mapsto \alpha - \beta$ . The third map is onto since the elements of the partition of the unity are *controlled functions*, id est, elements of  $\Omega_0^0(-)$  (cf. 3.2). Thus, the sequence is exact. This result is not longer true for more general coverings.

We shall use in this work the two following local calculations (see [11, Proposition 3.5.1 and Proposition 3.5.2] for the proofs).

**Proposition 3** *Let  $(\mathbb{R}^k, \mathcal{H})$  be a simple foliation. Consider  $\overline{p}$  a perversity on  $M$  and define the perversity  $\overline{p}$  on  $\mathbb{R}^k \times M$  by  $\overline{p}(\mathbb{R}^k \times S) = \overline{p}(S)$ . The canonical projection  $\text{pr} : \mathbb{R}^k \times M \rightarrow M$  induces the isomorphism*

$$\mathbb{H}_{\overline{p}}^*(M/\mathcal{F}) \cong \mathbb{H}_{\overline{p}}^*(\mathbb{R}^k \times M/\mathcal{H} \times \mathcal{F}).$$

**Proposition 4** *Let  $\mathcal{G}$  be a conical foliation without 0-dimensional leaves on the sphere  $\mathbb{S}^n$ . A perversity  $\overline{p}$  on  $c\mathbb{S}^n$  gives the perversity  $\overline{p}$  on  $\mathbb{S}^n$  defined by  $\overline{p}(S) = \overline{p}(S \times ]0, 1[)$ . The canonical projection  $\text{pr} : \mathbb{S}^n \times ]0, 1[ \rightarrow \mathbb{S}^n$  induces the isomorphism*

$$\mathbb{H}_{\overline{p}}^i(c\mathbb{S}^n/c\mathcal{G}) = \begin{cases} \mathbb{H}_{\overline{p}}^i(\mathbb{S}^n/\mathcal{G}) & \text{if } i \leq \overline{p}(\{\vartheta\}) \\ 0 & \text{if } i > \overline{p}(\{\vartheta\}). \end{cases}$$

In the next section we shall need the following technical Lemma.

**Lemma 1** *Let  $\Phi : K \times M \rightarrow M$  be a smooth action, where  $K$  is a compact Lie group, and let  $V$  be a fundamental vector field of this action. Consider a normal subgroup  $G$  of  $K$  and write  $\mathcal{F}$  the associated conical foliation on  $M$ . Then, the interior operator  $i_V : \Omega_{\overline{p}}^*(M/\mathcal{F}) \rightarrow \Omega_{\overline{p}}^{*-1}(M/\mathcal{F})$  is well defined, for any perversity  $\overline{p}$ .*

*Proof* Since the question is a local one, then it suffices to consider the case where  $M$  is a twisted product  $K \times_H N$ <sup>6</sup>. Notice that the blow up  $\Pi : K \times N \rightarrow K \times_H N$  is a  $K$ -equivariant map relatively to the action  $\ell \cdot (k, z) = (\ell \cdot k, z)$ . This gives  $\Pi_*(X^u, 0) = V$  for some  $u \in \mathfrak{k}$ . From Lemma 2 we know that it suffices to prove that the operator

$$i_{(X^u, 0)} : \Omega_{\overline{p}}^*(K \times N/\mathcal{K} \times \mathcal{N}) \longrightarrow \Omega_{\overline{p}}^{*-1}(K \times N/\mathcal{K} \times \mathcal{N})$$

is well defined. Since  $G \triangleleft K$  then the vector field  $X^u$  preserves the foliation  $\mathcal{K}$ . So, it suffices to prove that the operator

$$i_{(X^u, 0)} : \Omega_{\overline{p}}^*(K \times N) \longrightarrow \Omega_{\overline{p}}^{*-1}(K \times N)$$

<sup>6</sup> In fact,  $N$  is an euclidean space  $\mathbb{R}^a$  et  $\Theta$  is an orthogonal action.

is well defined. This comes from the fact that  $X^u$  acts on the  $K$ -factor while the perversion conditions are measured on the  $N$ -factor (cf. (1)).  $\square$

#### 4 The BIC of a twisted product

We compute now the BIC of a twisted product  $K \times_H N$  (cf. 2.3) for a perversity  $\bar{p}$  (cf. (1)).

**Lemma 2** *The natural projection  $\Pi: K \times N \rightarrow K \times_H N$  induces the differential monomorphism*

$$\Pi^*: \Omega_{\bar{p}}^*(K \times_H N / \mathcal{W}) \longrightarrow \Omega_{\bar{p}}^*(K \times N / K \times \mathcal{N}). \quad (2)$$

Moreover, given a differential form  $\omega$  on  $K \times_H N / \mathcal{W}$ , we have:

$$\Pi^* \omega \in \Omega_{\bar{p}}^*(K \times N / K \times \mathcal{N}) \iff \omega \in \Omega_{\bar{p}}^*(K \times_H N / \mathcal{W}). \quad (3)$$

*Proof* Notice that the injectivity of  $\Pi^*$  comes from the fact that  $\Pi$  is a surjection. For the rest, we proceed in several steps.

(a) *A foliated atlas for  $\pi: K \rightarrow K/H$ .*

Since  $\pi: K \rightarrow K/H$  is a  $H$ -principal bundle then it possesses an atlas  $\mathcal{A} = \left\{ \varphi: \pi^{-1}(U) \longrightarrow U \times H \right\}$  made up with  $H$ -equivariant charts:  $\varphi(k \cdot h^{-1}) = (\pi(k), h \cdot h_0)$  if  $\varphi(k) = (\pi(k), h_0)$ . We study the foliation  $\varphi_* \mathcal{K}$ . This equivariance property gives  $\varphi_* X_u = (0, Z^u)$  for each  $u \in \mathfrak{g} \cap \mathfrak{h}$ . Thus, the trace of the foliation  $\varphi_* \mathcal{K}$  on the fibers of the canonical projection  $\text{pr}: U \times H \rightarrow U$  is  $\mathcal{C}$ . On the other hand, since the map  $\pi$  is a  $G$ -equivariant map then  $\pi_* \mathcal{K} = \mathcal{D}$ , which gives  $\text{pr}_* \varphi_* \mathcal{K} = \mathcal{D}$ . We conclude that  $\varphi_* \mathcal{K} \subset \mathcal{D} \times \mathcal{C}$ . By dimension reasons we get  $\varphi_* \mathcal{K} = \mathcal{D} \times \mathcal{C}$ . The atlas  $\mathcal{A}$  is an  $H$ -equivariant foliated atlas of  $\pi$ .

(b) *A foliated atlas for  $\Pi: K \times N \rightarrow K \times_H N$ .*

We claim that  $\mathcal{A}_\# = \left\{ \bar{\varphi}: \pi^{-1}(U) \times_H N \longrightarrow U \times N / (U, \varphi) \in \mathcal{A} \right\}$  is a foliated atlas of  $K \times_H N$  where the map  $\bar{\varphi}$  is defined by  $\bar{\varphi}(< k, z >) = (\pi(k), (\Theta((\varphi^{-1}(\pi(k), e))^{-1} \cdot k, z)))$ . This map is a diffeomorphism whose inverse is  $\bar{\varphi}^{-1}(u, z) = < \varphi^{-1}(u, e), z >$ . It verifies

$$\bar{\varphi}_* \mathcal{W} \stackrel{2.3(a)}{=} \bar{\varphi}_* \Pi_*(\mathcal{K} \times \mathcal{I}) = \bar{\varphi}_* \Pi_*(\varphi^{-1} \times \mathbb{I}_N)_*(\mathcal{D} \times \mathcal{C} \times \mathcal{I}).$$

A straightforward calculation shows  $\bar{\varphi}_* \Pi_*(\varphi^{-1} \times \mathbb{I}_N) = (\mathbb{I}_U \times \Theta)$ . Since  $\mathcal{C}$  is defined by the action  $\Gamma$  then  $\Theta_*(\mathcal{C} \times \mathcal{I}) = \mathcal{N}$ . Finally we obtain  $\bar{\varphi}_* \mathcal{W} = \mathcal{D} \times \mathcal{N}$ .

(c) *Last Step.* Given  $(U, \varphi) \in \mathcal{A}_\#$ , we have the commutative diagram

$$\begin{array}{ccc} U \times H \times N & \xrightarrow{\varphi^{-1} \times \mathbb{I}_N} & K \times N \\ \downarrow Q & & \downarrow \Pi \\ U \times N & \xrightarrow{\bar{\varphi}^{-1}} & K \times_H N \end{array}$$

where  $Q(u, h, z) = (u, h^{-1} \cdot z)$ ,  $\Pi^{-1}(\text{Im } \bar{\varphi}^{-1}) = \text{Im } (\varphi^{-1} \times \mathbb{I}_N)$  and the rows are foliated imbeddings. Now, since (2) and (3) are local questions then it suffices to prove that

- $Q^*: \Omega_{\overline{p}}^*(U \times N/\mathcal{D} \times \mathcal{N}) \longrightarrow \Omega_{\overline{p}}^*(U \times H \times N/\mathcal{D} \times \mathcal{C} \times \mathcal{N})$  is well-defined, and
- $Q^*\omega \in \Omega_{\overline{p}}^*(U \times H \times N/\mathcal{D} \times \mathcal{C} \times \mathcal{N}) \iff \omega \in \Omega_{\overline{p}}^*(U \times N/\mathcal{D} \times \mathcal{N})$ , for any  $\omega \in \Omega^*(U \times R_{\mathcal{N}})$ .

This comes from the fact that the map

$$\nabla: (U \times H \times N, \mathcal{D} \times \mathcal{C} \times \mathcal{N}) \longrightarrow (U \times H \times N, \mathcal{D} \times \mathcal{C} \times \mathcal{N}),$$

defined by  $\nabla(u, h, z) = (u, h, h^{-1} \cdot z)$ , is a foliated diffeomorphism<sup>7</sup> and  $Q = \text{pr}_0 \circ \nabla$ , with  $\text{pr}_0: U \times H \times N \rightarrow U \times N$  canonical projection (cf. Proposition 3).  $\square$

#### 4.1 The Lie algebra $\mathfrak{k}$

We suppose in this paragraph that  $G$  is also dense on  $K$ . Choose  $\nu$  a bi-invariant riemannian metric on  $K$ , which exists by compactness. Consider

$$B = \{u_1, \dots, u_a, u_{a+1}, \dots, u_b, u_{b+1}, \dots, u_c, u_{c+1}, \dots, u_f\}$$

an orthonormal basis of the Lie algebra  $\mathfrak{k}$  of  $K$  with  $\{u_1, \dots, u_b\}$  basis of the Lie algebra  $\mathfrak{g}$  of  $G$  and  $\{u_{a+1}, \dots, u_c\}$  basis of the Lie algebra  $\mathfrak{h}$  of  $H$ . For each indice  $1 \leq i \leq f$  we shall write  $X_i \equiv X_{u_i}$  and  $X^i \equiv X^{u_i}$  (cf. 2.2).

Let  $\gamma_i \in \Omega^1(K)$  be the dual form of  $X_i$ , that is,  $\gamma_i = i_{X_i}\nu$ . Notice that  $\delta_{ij} = \gamma_j(X_i)$ . These forms are invariant by the left action of  $K$ . Since the flow of  $X^j$  is the multiplication on the left by  $\exp(tu_j)$  then  $L_{X^j}\gamma_i = 0$  for each  $1 \leq j \leq f$ .

For the differential, we have the formula  $d\gamma_l = \sum_{1 \leq i < j \leq f} C_{ij}^l \gamma_i \wedge \gamma_j$ , where  $[X_i, X_j]$  is  $\sum_{l=1}^f C_{ij}^l X_l$ , and  $1 \leq i, j, l \leq f$ . We have several restrictions on these coefficients. Since  $G \triangleleft K$  then  $\mathfrak{g}$  is an ideal of  $\mathfrak{k}$  and therefore we have

$$C_{ij}^l = 0 \text{ for } i \leq b < l.$$

Since  $K/G$  is an abelian group (cf. Proposition 2) then the induced bracket on  $\mathfrak{k}/\mathfrak{g}$  is zero and therefore we have

$$C_{ij}^l = 0 \text{ for } b < i, j, l \leq f.$$

These equations imply that

$$d\gamma_l = 0 \text{ for each } b < l. \quad (4)$$

The  $\mathcal{E}$ -basic differential forms in  $\bigwedge^*(\gamma_1, \dots, \gamma_f)$  are exactly  $\bigwedge^*(\gamma_{c+1}, \dots, \gamma_f)$  since they are cycles and the family  $\{X_1, \dots, X_c\}$  generates the foliation  $\mathcal{E}$ . This gives

$$H^*(K/\mathcal{E}) = \bigwedge^*(\gamma_{c+1}, \dots, \gamma_f). \quad (5)$$

#### 4.2 Two actions of $H/H_0$

The Lie group  $H$  preserves the foliation  $\mathcal{N}$  since the Lie group  $G \cap H$  is a normal subgroup of  $H$ . Put  $H_0$  the connected component of  $H$  containing the unity element. Since it is a connected compact Lie group then a standard argument shows that

$$\left(\mathbb{H}_{\overline{p}}^*(N/\mathcal{N})\right)^{H_0} = H^*\left(\left(\Omega_{\overline{p}}^*(N/\mathcal{N})\right)^{H_0}\right) = \mathbb{H}_{\overline{p}}^*(N/\mathcal{N}) \quad (6)$$

<sup>7</sup> Since  $G \cap H \triangleleft H$ .



(cf. [5, Theorem I, Ch. IV, vol. II]). We conclude that the finite group  $H/H_0$  acts naturally on  $\mathbb{H}_p^*(N/\mathcal{N})$ .

Since  $H_0$  is a connected Lie subgroup of  $GH$  then  $\left(H^*(K/\mathcal{E})\right)^{H_0} = H^*(K/\mathcal{E})$ . We conclude that the finite group  $H/H_0$  acts naturally on  $H^*(K/\mathcal{E})$ .

**Proposition 5** *Let  $(K, G, H)$  be a trio with  $G$  connected and dense in  $K$ . Then*

$$\mathbb{H}_p^*(K \times_H N/\mathcal{W}) = \left(H^*(K/\mathcal{E}) \otimes \mathbb{H}_p^*(N/\mathcal{N})\right)^{H/H_0}.$$

*Proof* Using the blow up  $\Pi: K \times N \rightarrow K \times_H N$ , the computation of  $\mathbb{H}_p^*(K \times_H N/\mathcal{W})$  can be done by using the complex  $\text{Im} \left\{ \Pi^*: \Omega_p^*(K \times_H N/\mathcal{F}) \rightarrow \Omega_p^*(K \times N/\mathcal{K} \times \mathcal{N}) \right\}$  (cf. Lemma 2). We study this complex in several steps. We fix  $B = \{u_1, \dots, u_f\}$  an orthonormal basis of  $\mathfrak{k}$  as in 4.1.

*(i) Description of  $\Omega^*(K \times R_{\mathcal{N}})$ .*

A differential form  $\omega \in \Omega^*(K \times R_{\mathcal{N}})$  is of the form

$$\eta + \sum_{1 \leq i_1 < \dots < i_\ell \leq f} \gamma_{i_1} \wedge \dots \wedge \gamma_{i_\ell} \wedge \eta_{i_1, \dots, i_\ell}, \quad (7)$$

where the forms  $\eta, \eta_{i_1, \dots, i_\ell} \in \Omega^*(K \times R_{\mathcal{N}})$  verify  $i_{X_j} \eta = i_{X_j} \eta_{i_1, \dots, i_\ell} = 0$  for each  $1 \leq j \leq f$  and each  $1 \leq i_1 < \dots < i_\ell \leq f$ .

*(ii) Description of  $\Pi_{\mathcal{K} \times \mathcal{N}}^*(K \times N)$ .*

Since the foliation  $\mathcal{K}$  is regular then we always can choose a conical chart of the form  $(U_1 \times U_2, \varphi_1 \times \varphi_2)$  where  $(U_1, \varphi_1)$  is a foliated chart of  $(K, \mathcal{K})$  and  $(U_2, \varphi_2)$  is a conical chart of  $(N, \mathcal{N})$ . The local blow up of the chart  $(U_1 \times U_2, \varphi_1 \times \varphi_2)$  is constructed from the second factor without modifying the first one. So, the differential forms  $\gamma_i$  are always perverse forms and a differential form  $\omega \in \Pi_{\mathcal{K} \times \mathcal{N}}^*(K \times N)$  is of the form (7) where  $\eta, \eta_{i_1, \dots, i_\ell} \in \Pi_{\mathcal{K} \times \mathcal{N}}^*(K \times N)$  verify  $i_{X_j} \eta = i_{X_j} \eta_{i_1, \dots, i_\ell} = 0$  for each  $1 \leq j \leq f$  and each  $1 \leq i_1 < \dots < i_\ell \leq f$ .

*(iii) Description of  $\Omega^*(K \times R_{\mathcal{N}}/\mathcal{K} \times \mathcal{N})$ .*

Take  $\omega \in \Omega^*(K \times R_{\mathcal{N}}/\mathcal{K} \times \mathcal{N})$ . Since  $\mathcal{K}$  is generated by the family  $\{X_j / 1 \leq j \leq b\}$  then  $L_{X_j} \omega = 0$  for any  $1 \leq j \leq b$ , or equivalently,  $R_g^* \omega = \omega$  for each  $g \in G$  since  $G$  is connected. By density,  $R_k^* \omega = \omega$  for each  $k \in K$  and therefore  $L_{X_j} \omega = 0$  for any  $1 \leq j \leq f$  since  $K$  is connected. We conclude that  $L_{X_j} \eta = L_{X_j} \eta_{i_1, \dots, i_\ell} = 0$  for any  $1 \leq j \leq f$  and each  $1 \leq i_1 < \dots < i_\ell \leq f$ . This gives  $\omega \in \bigwedge^*(\gamma_1, \dots, \gamma_f) \otimes \Omega^*(R_{\mathcal{N}})$ . The  $\mathcal{N}$ -basic differential forms of  $\Omega^*(R_{\mathcal{N}})$  are exactly  $\Omega^*(R_{\mathcal{N}}/\mathcal{N})$ . The  $\mathcal{K}$ -basic differential forms of  $\bigwedge^*(\gamma_1, \dots, \gamma_f)$  are exactly  $\bigwedge^*(\gamma_{b+1}, \dots, \gamma_f)$  (cf. (4)). From these two facts, we get

$$\Omega^*(K \times R_{\mathcal{N}}/\mathcal{K} \times \mathcal{N}) = \bigwedge^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega^*(R_{\mathcal{N}}/\mathcal{N})$$

as differential graduate commutative algebras.

*(iv) Description of  $\Omega_p^*(K \times N/\mathcal{K} \times \mathcal{N})$ .*

From (ii) and (iii) it suffices to control the perverse degree of the forms

$$\eta + \sum_{b+1 \leq i_1 < \dots < i_\ell \leq f} \gamma_{i_1} \wedge \dots \wedge \gamma_{i_\ell} \wedge \eta_{i_1, \dots, i_\ell} \in \bigwedge^*(\gamma_{b+1}, \dots, \gamma_f) \otimes \Pi_{\mathcal{N}}^*(N).$$

Consider  $S$  a stratum of  $\mathbf{S}_{\mathcal{N}}$ . From  $\|\gamma_i\|_{K \times S} = 0$  and  $\|\eta\|_{K \times S} = \|\eta\|_S$ , we get  $\|\gamma_{i_1} \wedge \dots \wedge \gamma_{i_\ell} \wedge \eta_{i_1, \dots, i_\ell}\|_{K \times S} = \|\eta_{i_1, \dots, i_\ell}\|_S$ . We conclude that

$$\Omega_{\overline{p}}^*(K \times N/\mathcal{K} \times \mathcal{N}) \cong \bigwedge^* (\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N})$$

(cf. 2.3 (b)).

$\langle v \rangle$  Description of  $\text{Im} \left\{ \Pi^*: \Omega_{\overline{p}}^*(K \times_H N/\mathcal{F}) \longrightarrow \Omega_{\overline{p}}^*(K \times N/\mathcal{K} \times \mathcal{N}) \right\}$ .

We denote by  $\{W_{a+1}, \dots, W_c\}$  the fundamental vector fields of the action  $\Theta: H \times N \rightarrow N$  associated to the basis  $\{u_{a+1}, \dots, u_c\}$ . Consider now the action  $\Upsilon: H \times (K \times N) \rightarrow (K \times N)$  defined by  $\Upsilon(h, (k, z)) = (k \cdot h^{-1}, \Theta(h, z))$ . Its fundamental vector fields associated to the basis  $\{u_{a+1}, \dots, u_c\}$  are  $\{(X_{a+1}, W_{a+1}), \dots, (X_c, W_c)\}$ . Given  $h \in H$ , we take  $\Upsilon_h: K \times N \rightarrow K \times N$  the map defined by  $\Upsilon_h(k, z) = \Upsilon(h, (k, z))$ . Then, we have

$$\text{Im } \Pi^* = \left\{ \omega \in \bigwedge^* (\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N}) \left/ \begin{array}{l} \text{(i) } i_{X_i} \omega = -i_{W_i} \omega \text{ if } a < i \leq c \\ \text{(ii) } L_{X_i} \omega = -L_{W_i} \omega \text{ if } a < i \leq c, \\ \text{(iii) } (\Upsilon_h)^* \omega = \omega \text{ for } h \in H. \end{array} \right. \right\}.$$

Let  $H_0$  be the unity connected component of  $H$ . Recall that the subgroup  $H_0$  is normal in  $H$  and that the quotient  $H/H_0$  is a finite group. Conditions (ii) gives that  $\omega$  is  $H_0$ -invariant. So, condition (iii) can be replaced by: (iv)  $(\Upsilon_h)^* \omega = \omega$  for  $h \in H/H_0$ . Therefore

$$\text{Im } \Pi^* = \left\{ \omega \in \bigwedge^* (\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N}) \left/ \begin{array}{l} \text{(i) } i_{X_i} \omega = -i_{W_i} \omega \text{ if } a < i \leq c \\ \text{(ii) } L_{X_i} \omega = -L_{W_i} \omega \text{ if } a < i \leq c. \end{array} \right. \right\}^{H/H_0}.$$

Since the group  $H/H_0$  is a finite one, we get that the cohomology  $H^*(\text{Im } \Pi^*)$  is isomorphic to  $(H^*(A^*))^{H/H_0}$ , where  $A^*$  is the differential complex

$$\left\{ \omega \in \bigwedge^* (\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N}) \left/ \begin{array}{l} \text{(i) } i_{X_i} \omega = -i_{W_i} \omega \text{ if } a < i \leq c \\ \text{(ii) } L_{X_i} \omega = -L_{W_i} \omega \text{ if } a < i \leq c \end{array} \right. \right\}.$$

So, it remains to compute  $H^*(A^*)$ . This computation can be simplified by using these three facts:

- $i_{W_i} \omega = L_{W_i} \omega = 0$  for each  $a < i \leq b$ , since the foliation  $\mathcal{N}$  is defined by the action of  $G \cap H$ .
- $i_{X_i} \gamma_j = \delta_{ij}$  for all  $i, j$  (cf. 4.1).
- $d\gamma_j = 0$  for  $b < j$  (cf. (4)).

We get that  $A^*$  is the differential complex

$$\left\{ \omega \in \bigwedge^* (\gamma_{b+1}, \dots, \gamma_f) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N}) \left/ \begin{array}{l} \text{(i) } i_{X_i} \omega = -i_{W_i} \omega \text{ if } b < i \leq c \\ \text{(ii) } 0 = L_{W_i} \omega \text{ if } b < i \leq c \end{array} \right. \right\}$$

$$= \bigwedge^* (\gamma_{c+1}, \dots, \gamma_f) \otimes \underbrace{\left\{ \omega \in \bigwedge^* (\gamma_{b+1}, \dots, \gamma_c) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N}) \left/ \begin{array}{l} \text{(i) } i_{X_i} \omega = -i_{W_i} \omega \text{ if } b < i \leq c \\ \text{(ii) } 0 = L_{W_i} \omega \text{ if } b < i \leq c \end{array} \right. \right\}}_{B^*}.$$

A straightforward computation gives that a form  $\omega \in \bigwedge^* (\gamma_{b+1}, \dots, \gamma_c) \otimes \Omega_{\overline{p}}^*(N/\mathcal{N})$  verifying (i) is in fact

$$\omega = \omega_0 + \sum_{b < i_1 < \dots < i_\ell \leq c} (-1)^\ell \gamma_{i_1} \wedge \dots \wedge \gamma_{i_\ell} \wedge (i_{W_{i_\ell}} \dots i_{W_{i_1}} \omega_0) \quad (8)$$

for some  $\omega_0 \in \Omega_{\overline{p}}^*(N/\mathcal{N})$  (cf. Lemma 1).

Consider now  $b < i, j \leq c$ . Since  $K/G$  is an abelian group (cf. Proposition 2) and  $H$  is a Lie group then  $[W_i, W_j] = \sum_{l=a+1}^b C_{ij}^l W_l$ . Then,  $i_{[W_i, W_j]} \omega_0 = 0$  since the foliation  $\mathcal{N}$  is defined by the action of  $G \cap H$ . So, the canonical writing of a form  $\omega \in B^*$  is (8) for some  $\omega_0 \in \left\{ \eta \in \Omega_{\overline{p}}^*(N/\mathcal{N}) / L_{W_i} \eta = 0 \text{ if } b < i \leq c \right\} = \left( \Omega_{\overline{p}}^*(N/\mathcal{N}) \right)^{H_0}$ .

Then, the operator  $\Delta: B^* \rightarrow \left( \Omega_{\overline{p}}^*(N/\mathcal{N}) \right)^{H_0}$ , defined by  $\Delta(\omega) = \omega_0$ , is a differential isomorphism. We conclude that the differential complex  $A^*$  is isomorphic to  $\bigwedge^* (\gamma_{c+1}, \dots, \gamma_f) \otimes \left( \Omega_{\overline{p}}^*(N/\mathcal{N}) \right)^{H_0}$  and therefore  $H^*(A) \cong H^*(K/\mathcal{E}) \otimes \mathbb{H}_{\overline{p}}^*(N/\mathcal{N})$  (cf. (5) and (6)). Since the operator  $\Delta$  is  $(H/H_0)$ -equivariant (cf. 4.2) then we get

$$\mathbb{H}_{\overline{p}}^*(K \times_H N/\mathcal{W}) = H^*(\text{Im } \Pi^*) = \left( H^*(A) \right)^{H/H_0} = \left( H^*(K/\mathcal{E}) \otimes \mathbb{H}_{\overline{p}}^*(N/\mathcal{N}) \right)^{H/H_0}.$$

This ends the proof.  $\square$

### 4.3 Remarks

(a) When the Lie group  $G$  is commutative then  $K$  is also commutative. Differential forms  $\gamma_\bullet$  are  $K$ -invariants on the left and on the right, so  $\left( H^*(K/\mathcal{E}) \right)^H = H^*(K/\mathcal{E})$  and therefore

$$\mathbb{H}_{\overline{p}}^*(K \times_H N/\mathcal{W}) = H^*(K/\mathcal{E}) \otimes \left( \mathbb{H}_{\overline{p}}^*(N/\mathcal{N}) \right)^{H/H_0} = H^*(K/\mathcal{E}) \otimes \left( \mathbb{H}_{\overline{p}}^*(N/\mathcal{N}) \right)^H$$

as it has been proved in [11, Proposition 3.8.4].

(b) Since the foliation  $\mathcal{E}$  is a riemannian foliation defined on a compact manifold then we know that the cohomology  $H^*(K/\mathcal{E})$  is finite (cf. [4]). So, the finiteness of  $\mathbb{H}_{\overline{p}}^*(K \times_H N/\mathcal{W})$  depends on the finiteness of  $\mathbb{H}_{\overline{p}}^*(N/\mathcal{N})$ .

## 5 Finiteness of the BIC

We prove in this section that the BIC of a Killing foliation on a compact manifold is finite dimensional. First of all, we present two geometrical tools we shall use in the proof: the isotropy type stratification and the Molino's blow up.

We fix an isometric action  $\Phi: G \times M \rightarrow M$  on the compact manifold  $M$ . We denote by  $\mathcal{F}$  the induced Killing foliation. For the study of  $\mathcal{F}$  we can suppose that  $G$  is connected (see Lemma 1). We fix  $K$  a tamer group. Notice that the group  $G$  is normal in  $K$  and the quotient  $K/G$  is commutative (cf. Proposition 2).

### 5.1 Isotropy type stratification

The isotropy type stratification  $\mathbf{S}_{K,M}$  of  $M$  is defined by the equivalence relation<sup>8</sup>:

$$x \sim y \Leftrightarrow K_x \text{ is conjugated to } K_y.$$

When depth  $\mathbf{S}_{K,M} > 0$ , any closed stratum  $S \in \mathbf{S}_{K,M}$  is a  $K$ -invariant submanifold of  $M$  and then it possesses a  $K$ -invariant tubular neighborhood  $(T, \tau, S, \mathbb{R}^m)$  whose structural group is  $O(m)$ . Recall that there are the following smooth maps associated with this neighborhood:

- + The *radius map*  $\rho: T \rightarrow [0, 1[$  defined fiberwise from the assignation  $[x, t] \mapsto t$ . Each  $t \neq 0$  is a regular value of the  $\rho$ . The pre-image  $\rho^{-1}(0)$  is  $S$ . This map is  $K$ -invariant, that is,  $\rho(k \cdot z) = \rho(z)$ .
- + The *contraction*  $H: T \times [0, 1] \rightarrow T$  defined fiberwisely from  $([x, t], r) \mapsto [x, rt]$ . The restriction  $H_t: T \rightarrow T$  is an embedding for each  $t \neq 0$  and  $H_0 \equiv \tau$ . We shall write  $H(z, t) = t \cdot z$ . This map is  $K$ -invariant, that is,  $t \cdot (k \cdot z) = k \cdot (t \cdot z)$ .

The hyper-surface  $D = \rho^{-1}(1/2)$  is the *tube* of the tubular neighborhood. It is a  $K$ -invariant submanifold of  $T$ . Notice that the map

$$\nabla: D \times [0, 1[ \longrightarrow T,$$

defined by  $\nabla(z, t) = (2t) \cdot z$  is a  $K$ -equivariant smooth map, where  $K$  acts trivially on the  $[0, 1[$ -factor. Its restriction  $\nabla: D \times ]0, 1[ \longrightarrow T \setminus S$  is a  $K$ -equivariant diffeomorphism.

Denote  $S_{min}$  the union of closed (minimal) strata and choose  $T_{min}$  a disjoint family of  $K$ -invariant tubular neighborhoods of the closed strata. The union of associated tubes is denoted by  $D_{min}$ . Notice that the induced map  $\nabla_{min}: D_{min} \times ]0, 1[ \longrightarrow T_{min} \setminus S_{min}$  is a  $K$ -equivariant diffeomorphism.

### 5.2 Molino's blow up

The Molino' blow up [7] of the foliation  $\mathcal{F}$  produces a new foliation  $\widehat{\mathcal{F}}$  of the same kind but of smaller depth. We suppose depth  $\mathbf{S}_{K,M} > 0$ . The *blow up* of  $M$  is the compact manifold

$$\widehat{M} = \left\{ \left( D_{min} \times ]-1, 1[ \right) \coprod \left( (M \setminus S_{min}) \times \{-1, 1\} \right) \right\} / \sim,$$

where  $(z, t) \sim (\nabla_{min}(z, |t|), t/|t|)$ , and the map  $\mathcal{L}: \widehat{M} \longrightarrow M$  defined by

$$\mathcal{L}(v) = \begin{cases} \nabla_{min}(z, |t|) & \text{if } v = (z, t) \in D_{min} \times ]-1, 1[ \\ z & \text{if } v = (z, j) \in (M \setminus S_{min}) \times \{-1, 1\}. \end{cases}$$

<sup>8</sup> For notions related with compact Lie group actions, we refer the reader to [1].

Notice that  $\mathcal{L}$  is a continuous map whose restriction  $\mathcal{L}: \widehat{M} \setminus \mathcal{L}^{-1}(S_{\min}) \rightarrow M \setminus S_{\min}$  is a  $K$ -equivariant smooth trivial 2-covering.

Since the map  $\nabla_{\min}$  is  $K$ -equivariant then  $\Phi$  induces the action  $\widehat{\Phi}: K \times \widehat{M} \rightarrow \widehat{M}$  by saying that the blow-up  $\mathcal{L}$  is  $K$ -equivariant. The open submanifolds  $\mathcal{L}^{-1}(T_{\min})$  and  $\mathcal{L}^{-1}(T_{\min} \setminus S_{\min})$  are clearly  $K$ -diffeomorphic to  $D_{\min} \times ]-1, 1[$  and  $D_{\min} \times (]-1, 0[ \cup ]0, 1[)$  respectively.

The restriction  $\widehat{\Phi}: G \times \widehat{M} \rightarrow \widehat{M}$  is an isometric action with  $K$  as a tamer group. The induced Killing foliation is  $\widehat{\mathcal{F}}$ . Foliations  $\mathcal{F}$  and  $\widehat{\mathcal{F}}$  are related by  $\mathcal{L}$  which is a foliated map. Moreover, if  $S$  is a not minimal stratum of  $\mathbf{S}_{K,M}$  then there exists a unique stratum  $S' \in \mathbf{S}_{K,\widehat{M}}$  such that  $\mathcal{L}^{-1}(S) \subset S'$ . The family  $\{S' / S \in \mathbf{S}_{K,M}\}$  covers  $\widehat{M}$  and verifies the relationship:  $S_1 \prec S_2 \Leftrightarrow S'_1 \prec S'_2$ . We conclude the important property

$$\text{depth } \mathbf{S}_{K,\widehat{M}} < \text{depth } \mathbf{S}_{K,M}. \quad (9)$$

### 5.3 Finiteness of a tubular neighborhood

We suppose  $\text{depth } \mathbf{S}_{K,M} > 0$ . Consider a closed stratum  $S \in \mathbf{S}_{K,M}$ . Take  $(T, \tau, S, \mathbb{R}^m)$  a  $K$ -invariant tubular neighborhood. We fix a base point  $x \in S$ . The isotropy subgroup  $K_x$  acts orthogonally on the fiber  $\mathbb{R}^m = \tau^{-1}(x)$ . So, the induced action  $\Lambda_x: G_x \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  is an isometric action, it gives the Killing foliation  $\mathcal{N}$  on  $\mathbb{R}^m$ .

**Proposition 6** *If the BIC of  $(\mathbb{R}^m, \mathcal{N})$  is finite dimensional then the BIC of  $(T, \mathcal{F})$  is also finite dimensional.*

*Proof* We proceed in two steps.

(a)  $K_y = K_x$  for each  $y \in S$ .

The canonical projection  $\pi: S \rightarrow S/K$  is an homogeneous bundle with fiber  $K/K_x$ . For any open subset  $V \subset S/K$  the pull back  $\tau^{-1}\pi^{-1}(V)$  is a  $K$ -invariant subset of  $T$ , then we can apply the Mayer-Vietoris technics to this kind of subsets (cf. 3.5).

Since the manifold  $S/K$  is a compact one then we can find a finite good covering  $\{U_i / i \in I\}$  of it (cf. [2]). An inductive argument on the cardinality of  $I$  reduces the proof of the Lemma to the case where  $T = \tau^{-1}\pi^{-1}(V)$ , where  $V$  is a contractible open subset of  $S/K$ .

Here, the manifold  $T$  is  $K$ -equivalently diffeomorphic to  $V \times (K \times_{K_x} \mathbb{R}^m)$ , where  $K$  does not act on the first factor. So, the natural retraction of  $V$  to a point gives a  $K$ -equivariant retraction of  $T$  to the twisted product  $K \times_{K_x} \mathbb{R}^m$ . Now the result comes directly from 4.3(b) since  $(K, G, K_x)$  is a trio.

(b) *General case.*

The stratum  $S$  is  $K$ -equivariantly diffeomorphic to the twisted product  $K \times_{N(K_x)} F$  where  $N(K_x)$  is the normalizer of  $K_x$  on  $K$  and  $F = S^{K_x}$ . So, the tubular neighborhood  $T$  is  $K$ -equivariantly diffeomorphic to the twisted product  $K \times_{N(H)} N$  where  $N$  is the manifold  $\tau^{-1}(F)$ . The previous case gives that the BIC of  $(N, \mathcal{F}_N)$  is finite dimensional. Now the result comes directly from 4.3(b) since  $(K, G, N(K_x))$  is a trio.  $\square$

The main result of this work is the following

**Theorem 1** *The BIC of the foliation determined by an isometric action on a compact manifold is finite dimensional.*

*Proof* Let  $\mathcal{F}$  be a Killing foliation defined on a compact manifold  $M$  induced by an isometric action  $\Phi: G \times M \rightarrow M$  where  $G$  is a Lie group. Without loss of generality we can suppose that the Lie group  $G$  is a connected one (cf. Lemma 1). We fix a tamer group  $K$ . We know that  $G$  is normal in  $K$  and the quotient group  $K/G$  is commutative (cf. Proposition 2).

Let us consider the following statement

$$\mathfrak{A}(U, \mathcal{F}) = \text{“The BIC } \mathbb{H}_p^*(U/\mathcal{F}) \text{ is finite dimensional for each perversity } \bar{p}\text{”}$$

where  $U \subset M$  is a  $K$ -invariant submanifold. We prove  $\mathfrak{A}(M, \mathcal{F})$  by induction on  $\dim M$ . The result is clear when  $\dim M = 0$ . We suppose  $\mathfrak{A}(W, \mathcal{F})$  for any  $K$ -invariant compact submanifold  $W$  of  $M$  with  $\dim W < \dim M$  and we prove  $\mathfrak{A}(M, \mathcal{F})$ . We proceed in several steps.

*First step: 0-depth.* Let us suppose  $\text{depth } \mathbf{S}_{K,M} = 0$ . Since  $G \triangleleft K$  and  $K_x$  is conjugated to  $K_y$  then  $G_x$  is conjugated to  $G_y$ ,  $\forall x, y \in M$ . We get that the foliation  $\mathcal{F}$  is a (regular) riemannian foliation (cf. [7]). Its BIC is just the basic cohomology (cf. 3.3). Then  $\mathfrak{A}(M, \mathcal{F})$  comes from [4].

*Second step: Inside  $M$ .* Let us suppose  $\text{depth } \mathbf{S}_{K,M} > 0$ . The family  $\{M \setminus S_{\min}, T_{\min}\}$  is a basic covering of  $M$  and the we get the exact sequence (cf. 3.5)

$$0 \rightarrow \Omega_{\bar{p}}^*(M/\mathcal{F}) \rightarrow \Omega_{\bar{p}}^*((M \setminus S_{\min})/\mathcal{F}) \oplus \Omega_{\bar{p}}^*(T_{\min}/\mathcal{F}) \rightarrow \Omega_{\bar{p}}^*((T_{\min} \setminus S_{\min})/\mathcal{F}) \rightarrow 0.$$

The Five Lemma gives

$$\mathfrak{A}(T_{\min} \setminus S_{\min}, \mathcal{F}), \mathfrak{A}(T_{\min}, \mathcal{F}) \text{ and } \mathfrak{A}(M \setminus S_{\min}, \mathcal{F}) \implies \mathfrak{A}(M, \mathcal{F}).$$

Since  $T_{\min} \setminus S_{\min}$  is  $K$ -diffeomorphic to  $D_{\min} \times ]0, 1[$  (cf. (cf. 5.1)) then  $\mathfrak{A}(D_{\min}, \mathcal{F}) \implies \mathfrak{A}(T_{\min} \setminus S_{\min}, \mathcal{F})$ . The inequality  $\dim D_{\min} < \dim M$  gives

$$\mathfrak{A}(T_{\min}, \mathcal{F}) \text{ and } \mathfrak{A}(M \setminus S_{\min}, \mathcal{F}) \implies \mathfrak{A}(M, \mathcal{F}).$$

In order to prove  $\mathfrak{A}(T_{\min}, \mathcal{F})$  it suffices to prove  $\mathfrak{A}(T, \mathcal{F})$  where  $(T, \tau, S, \mathbb{R}^m)$  a  $K$ -invariant tubular neighborhood of closed stratum  $S$  of  $\mathbf{S}_{K,M}$ . Following Proposition 6 we have

$$\mathfrak{A}(\mathbb{R}^m, \mathcal{N}) \implies \mathfrak{A}(T, \mathcal{F}) \implies \mathfrak{A}(T_{\min}, \mathcal{F}).$$

Consider the orthogonal decomposition  $\mathbb{R}^m = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ , where  $\mathbb{R}^{m_1} = (\mathbb{R}^m)^{G_x}$ . The only fixed point of the restriction  $A_x: G_x \times \mathbb{R}^{m_2} \rightarrow \mathbb{R}^{m_2}$  is the origin. So, there exists a Killing foliation<sup>9</sup>  $\mathcal{G}$  on the sphere  $\mathbb{S}^{m_2-1}$  with  $(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}, \mathcal{F}) = (\mathbb{R}^{m_1} \times c\mathbb{S}^{m_2-1}, \mathcal{I} \times c\mathcal{G})$ . Propositions 3 and 4 give:

$$\mathfrak{A}(\mathbb{S}^{m_2-1}, \mathcal{G}) \implies \mathfrak{A}(\mathbb{R}^{m_1} \times c\mathbb{S}^{m_2-1}, \mathcal{I} \times c\mathcal{G}) \implies \mathfrak{A}(\mathbb{R}^m, \mathcal{N}).$$

Finally, since  $\dim \mathbb{S}^{m_2-1} < m \leq \dim T \leq \dim M$  we have

$$\mathfrak{A}(M \setminus S_{\min}, \mathcal{F}) \implies \mathfrak{A}(M, \mathcal{F}). \quad (10)$$

*Third step: Blow-up.* Let us suppose  $\text{depth } \mathbf{S}_{K,M} > 0$ . The family  $\{\mathcal{L}^{-1}(M \setminus S_{\min}), \mathcal{L}^{-1}(T_{\min})\}$  is a basic covering of  $\widehat{M}$  and the we get the exact sequence (cf. 3.5)

$$0 \rightarrow \Omega_{\bar{p}}^*(\widehat{M}/\widehat{\mathcal{F}}) \rightarrow \Omega_{\bar{p}}^*(\mathcal{L}^{-1}(M \setminus S_{\min})/\widehat{\mathcal{F}}) \oplus \Omega_{\bar{p}}^*(\mathcal{L}^{-1}(T_{\min})/\widehat{\mathcal{F}}) \rightarrow \Omega_{\bar{p}}^*(\mathcal{L}^{-1}(T_{\min} \setminus S_{\min})/\widehat{\mathcal{F}}) \rightarrow 0.$$

Following 5.2 we have that

<sup>9</sup> It is given by the orthogonal action  $A_x: G_x \times \mathbb{S}^{m_2-1} \rightarrow \mathbb{S}^{m_2-1}$ .

- $\mathcal{L}^{-1}(M \setminus S_{\min})$  is  $K$ -diffeomorphic to two copies of  $M \setminus S_{\min}$ ,
- $\mathcal{L}^{-1}(T_{\min})$  is  $K$ -diffeomorphic to  $D_{\min} \times ]-1, 1[$ ,
- $\mathcal{L}^{-1}(T_{\min} \setminus S_{\min})$  is  $K$ -diffeomorphic to  $D_{\min} \times (]-1, 0[ \cup ]0, 1[)$ .

Now, the Five Lemma gives

$$\mathfrak{A}(D_{\min}, \widehat{\mathcal{F}}) \text{ and } \mathfrak{A}(\widehat{M}, \widehat{\mathcal{F}}) \implies \mathfrak{A}(M \setminus S_{\min}, \mathcal{F}).$$

But, the inequality  $\dim D_{\min} < \dim M$  gives

$$\mathfrak{A}(\widehat{M}, \widehat{\mathcal{F}}) \implies \mathfrak{A}(M \setminus S_{\min}, \mathcal{F}). \quad (11)$$

*Forth step: Final blow-up.* When  $\text{depth } S_{K,M} = 0$  we get  $\mathfrak{A}(M, \mathcal{F})$  from the First step. Let us suppose  $\text{depth } S_{K,M} > 0$ . From (10) and (11) we get

$$\mathfrak{A}(\widehat{M}, \widehat{\mathcal{F}}) \implies \mathfrak{A}(M, \mathcal{F}).$$

with depth  $S_{K,\widehat{M}} < \text{depth } S_{K,M}$  (cf. (9)). By iterating this procedure we get

$$\mathfrak{A}(\widetilde{M}, \widetilde{\mathcal{F}}) = \mathfrak{A}\left(\widehat{\widetilde{M}}, \widehat{\widetilde{\mathcal{F}}}\right) \implies \dots \implies \mathfrak{A}(\widehat{M}, \widehat{\mathcal{F}}) \implies \mathfrak{A}(M, \mathcal{F}),$$

with depth  $S_{K,\widehat{M}} = 0$ . We finish the proof by applying again the First Step.  $\square$

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